



Multiple positive solutions for discrete nonlocal boundary value problems

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Abstract

In this paper, we investigate a second-order nonlinear difference equation with sign-changing nonlinearity subject to two different sets of nonlocal boundary conditions. The explicit expressions of the associated Green's functions are presented. By using a recently developed fixed point theorem, we establish sufficient conditions for the existence of multiple positive solutions of the boundary value problem.

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1. Introduction

In this paper, we consider the second-order difference equation

$$\nabla \Delta u(k) + f(k, u(k)) = 0, \quad k \in \mathbb{Z}_{1,T}, \quad (1.1)$$

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subject to one of the following discrete nonlocal boundary conditions:

$$(i) \quad u(0) - \beta \Delta u(0) = 0, \quad u(T+1) = \alpha u(l), \quad (1.2)$$

or

$$(ii) \quad \Delta u(0) = 0, \quad u(T+1) = \alpha u(l). \quad (1.3)$$

Here $T \in \{4, 5, \dots\}$ is fixed, $0 < \alpha < 1$, $\beta > 0$, $l \in \mathbb{Z}_{2,T-1}$, and $f: \mathbb{Z}_{0,T+1} \times [0, \infty) \rightarrow \mathbb{R}$ is continuous, where for any $a, b \in \mathbb{Z}$ with $a \leq b$, we denote by $\mathbb{Z}_{a,b} := [a, b] \cap \mathbb{Z}$.

Positive solutions for discrete boundary value problems, in its obvious applicable context, attract much attention in recent years, for example, see [1,3–7,10,13–18]. Most results so far in this context have been obtained mainly by using the fixed point theorem in cones, such as Krasnoselskii fixed point theorem [11], Leggett–Williams theorem [12], Avery and Henderson's theorem [2], and so on. In order to apply the concavity of solutions in the proofs, all the aforesaid works in [3–7,9,13–16] were done under the assumption that the nonlinear term is nonnegative. In this paper, by using a recently obtained fixed point theorem [8], we obtain sufficient conditions for the existence of multiple positive solutions for the BVP $\{(1.1), (1.2)\}$ and the BVP $\{(1.1), (1.3)\}$ for the case where the nonlinear term f can change sign. In this way we remove the usual restriction on the sign of f .

We shall first state the fixed point theorem which is our main tool. For a cone K in a Banach space X with norm $\|\cdot\|$ and a constant $r > 0$, let $K_r = \{x \in K: \|x\| < r\}$, $\partial K_r = \{x \in K: \|x\| = r\}$. Suppose $\varphi: K \rightarrow \mathbb{R}^+$ is a continuously increasing functional, i.e., φ is continuous and $\varphi(\lambda x) \leq \varphi(x)$ for $\lambda \in (0, 1)$. Let

$$K(b) = \{x \in K: \varphi(x) < b\},$$

$$\partial K(b) = \{x \in K: \varphi(x) = b\}.$$

and

$$K_a(b) = \{x \in K: \|x\| > a, \varphi(x) < b\}.$$

Theorem A. [8] *Let X be a real Banach space with norm $\|\cdot\|$ and $K, K' \subset X$ be two cones with $K' \subset K$. Suppose $\Phi: K \rightarrow K$ and $\Phi': K' \rightarrow K'$ are two completely continuous operators and $\varphi: K' \rightarrow \mathbb{R}^+$ is a continuously increasing functional satisfying $\varphi(x) \leq \|x\| \leq M\varphi(x)$ for all x in K' , where $M \geq 1$ is a constant. If there are constants $b > a > 0$ such that*

$$(C1) \quad \|\Phi x\| < a \text{ for } x \in \partial K_a,$$

$$(C2) \quad \|\Phi' x\| < a \text{ for } x \in \partial K'_a \text{ and } \varphi(\Phi' x) > b \text{ for } x \in \partial K'(b),$$

$$(C3) \quad \Phi x = \Phi' x \text{ for } x \in K'_a(b) \cap \{u: \Phi' u = u\},$$

then Φ has in K at least two fixed points u_1 and u_2 such that

$$0 \leq \|u_1\| < a < \|u_2\|, \quad \varphi(u_2) < b.$$

Let $C(\mathbb{Z}_{0,T+1}) = C(\mathbb{Z}_{0,T+1}, \mathbb{R})$ denote the class of real valued functions ω on $\mathbb{Z}_{0,T+1}$ with norm $\|\omega\| = \max_{k \in \mathbb{Z}_{0,T+1}} |\omega(k)|$. Observe that $C(\mathbb{Z}_{0,T+1})$ is a Banach space.

Remark 1.1. As suggested by the notation, by equipping $\mathbb{Z}_{0,T+1}$ with the discrete topology, every $\omega \in C(\mathbb{Z}_{0,T+1})$ is continuous.

We shall discuss BVP $\{(1.1), (1.2)\}$ and $\{(1.1), (1.3)\}$ separately.

2. Positive solutions to BVP {(1.1), (1.2)}

Lemma 2.1. Suppose $T \in \{4, 5, \dots\}$, $\ell \in \mathbb{Z}_{2,T-1}$, and $\alpha, \beta \in \mathbb{R}$ are real numbers with $\beta \neq -1$ and $(T+1-\alpha l) + \beta(1-\alpha) \neq 0$. For any $y \in C(\mathbb{Z}_{1,T})$, the BVP

$$\begin{cases} \nabla \Delta u(k) + y(k) = 0, & k \in \mathbb{Z}_{1,T}, \\ u(0) - \beta \Delta u(0) = 0, & u(T+1) = \alpha u(l) \end{cases} \quad (2.1)$$

has a unique solution

$$\begin{aligned} u(k) = & - \sum_{j=1}^{k-1} (k-j)y(j) + \sum_{j=1}^T \frac{k+\beta}{(T+1-\alpha l) + \beta(1-\alpha)} (T+1-j)y(j) \\ & - \sum_{j=1}^{l-1} \frac{\alpha(k+\beta)}{(T+1-\alpha l) + \beta(1-\alpha)} (l-j)y(j), \quad k \in \mathbb{Z}_{0,T+1}. \end{aligned} \quad (2.3)$$

Here for the sake of convenience, we define $\sum_{i=m_1}^{m_2} f(i) = 0$ for $m_2 < m_1$.

Proof. From (2.1) we have, for any $i \in \mathbb{Z}_{1,T}$,

$$\Delta u(i) - \Delta u(i-1) = -y(i).$$

Then for any $j \in \mathbb{Z}_{1,T}$,

$$\Delta u(j) - \Delta u(0) = \sum_{i=1}^j [\Delta u(i) - \Delta u(i-1)] = - \sum_{i=1}^j y(i),$$

i.e.,

$$\Delta u(j) - u(1) + u(0) = - \sum_{i=1}^j y(i).$$

From boundary condition $u(0) - \beta \Delta u(0) = 0$, we have

$$u(0) - \beta u(1) + \beta u(0) = 0,$$

thus

$$u(0) = \frac{\beta u(1)}{1+\beta}.$$

So

$$\Delta u(j) = - \sum_{i=1}^j y(i) + \frac{u(1)}{1+\beta}.$$

Then

$$\begin{aligned} u(k) &= \sum_{j=1}^{k-1} \Delta u(j) + u(1) = - \sum_{j=1}^{k-1} \sum_{i=1}^j y(i) + \sum_{j=1}^{k-1} \frac{u(1)}{1+\beta} + u(1) \\ &= - \sum_{j=1}^{k-1} \sum_{i=1}^j y(i) + \frac{k+\beta}{1+\beta} u(1) = - \sum_{j=1}^{k-1} (k-j)y(j) + \frac{k+\beta}{1+\beta} u(1). \end{aligned}$$

Together with the boundary condition $u(T+1) = \alpha u(l)$, we get

$$u(1) = \sum_{j=1}^T \frac{1+\beta}{(T+1-\alpha l) + \beta(1-\alpha)} (T+1-j)y(j) - \sum_{j=1}^{l-1} \frac{\alpha(1+\beta)}{(T+1-\alpha l) + \beta(1-\alpha)} (l-j)y(j).$$

So for all $k \in \mathbb{Z}_{0,T+1}$,

$$u(k) = - \sum_{j=1}^{k-1} (k-j)y(j) + \sum_{j=1}^T \frac{(k+\beta)(T+1-j)y(j)}{(T+1-\alpha l) + \beta(1-\alpha)} - \sum_{j=1}^{l-1} \frac{\alpha(k+\beta)(l-j)y(j)}{(T+1-\alpha l) + \beta(1-\alpha)}. \quad \square$$

Lemma 2.2. If $T \in \{4, 5, \dots\}$, $l \in \mathbb{Z}_{2,T-1}$, and $\alpha, \beta \in \mathbb{R}$ are real numbers with $\beta \neq -1$ and $(T+1-\alpha l) + \beta(1-\alpha) \neq 0$, the Green's function for the BVP $\{(2.1), (2.2)\}$ is given by

$$G(k, j) = \begin{cases} \frac{(j+\beta)[T+1-k-\alpha(l-k)]}{(T+1-\alpha l) + \beta(1-\alpha)}, & j < k, \quad j < l, \\ \frac{(T+1-k)(j+\beta) + \alpha(l+\beta)(k-j)}{(T+1-\alpha l) + \beta(1-\alpha)}, & l \leq j < k, \\ \frac{(k+\beta)[T+1-j-\alpha(l-j)]}{(T+1-\alpha l) + \beta(1-\alpha)}, & k \leq j < l, \\ \frac{(k+\beta)(T+1-j)}{(T+1-\alpha l) + \beta(1-\alpha)}, & j \geq k, \quad j \geq l. \end{cases} \quad (2.4)$$

Proof. If $k \geq l$, the unique solution (2.3) can be written as

$$u(k) = - \sum_{j=1}^{l-1} (k-j)y(j) - \sum_{j=l}^{k-1} (k-j)y(j) + \sum_{j=1}^{l-1} \frac{k+\beta}{(T+1-\alpha l) + \beta(1-\alpha)} (T+1-j)y(j) + \sum_{j=l}^{k-1} \frac{k+\beta}{(T+1-\alpha l) + \beta(1-\alpha)} (T+1-j)y(j) + \sum_{j=k}^T \frac{k+\beta}{(T+1-\alpha l) + \beta(1-\alpha)} (T+1-j)y(j) - \sum_{j=1}^{l-1} \frac{\alpha(k+\beta)}{(T+1-\alpha l) + \beta(1-\alpha)} (l-j)y(j)$$

$$\begin{aligned}
&= \sum_{j=1}^{l-1} \frac{(j+\beta)[T+1-k-\alpha(l-k)]}{(T+1-\alpha l)+\beta(1-\alpha)} y(j) \\
&\quad + \sum_{j=l}^{k-1} \frac{(T+1-k)(j+\beta)+\alpha(k-j)(l+\beta)}{(T+1-\alpha l)+\beta(1-\alpha)} y(j) \\
&\quad + \sum_{j=k}^T \frac{(k+\beta)(T+1-j)}{(T+1-\alpha l)+\beta(1-\alpha)} y(j).
\end{aligned}$$

Similarly, if $k < l$, the unique solution (2.3) can be written as

$$\begin{aligned}
u(k) &= \sum_{j=1}^{k-1} \frac{(j+\beta)[T+1-k-\alpha(l-k)]}{(T+1-\alpha l)+\beta(1-\alpha)} y(j) \\
&\quad + \sum_{j=k}^{l-1} \frac{(k+\beta)[T+1-j-\alpha(l-j)]}{(T+1-\alpha l)+\beta(1-\alpha)} y(j) \\
&\quad + \sum_{j=l}^T \frac{(k+\beta)(T+1-j)}{(T+1-\alpha l)+\beta(1-\alpha)} y(j).
\end{aligned}$$

Together with the fact that the unique solution of $\{(2.1), (2.2)\}$ can be written as $u(k) = \sum_{j=1}^T G(k, j)y(j)$, we have Lemma 2.2. \square

By our basic assumption, $l \in \mathbb{Z}_{2, T-1}$, $0 < \alpha < 1$ and $\beta > 0$, it is clear that the Green's function $G(k, j)$ for the BVP $\{(2.1), (2.2)\}$ as given in (2.4) satisfies $G(k, j) > 0$ for $k \in \mathbb{Z}_{0, T+1}$, $j \in \mathbb{Z}_{1, T}$.

By (2.3), letting $y(k) \equiv 1$, we have

$$\sum_{j=1}^T G(k, j) = \begin{cases} -\frac{k(k-1)}{2} + \frac{(k+\beta)[T(T+1)-\alpha l(l-1)]}{2[(T+1-\alpha l)+\beta(1-\alpha)]}, & k \in \mathbb{Z}_{1, T+1}, \\ \frac{\beta[T(T+1)-\alpha l(l-1)]}{2[(T+1-\alpha l)+\beta(1-\alpha)]}, & k = 0. \end{cases} \quad (2.5)$$

Let

$$M = \max_{k \in \mathbb{Z}_{0, T+1}} \sum_{j=1}^T G(k, j), \quad m = \min_{k \in \{h, \dots, T+1-h\}} \sum_{j=h}^{T+1-h} G(k, j),$$

here h is a fixed integer in $\{2, \dots, [\frac{T}{2}]\}$. It is clear that $0 < m < M$.

Let $X = C(\mathbb{Z}_{0, T+1})$, $K = \{u \in X: u(k) \geq 0, k \in \mathbb{Z}_{0, T+1}\}$, $K' = \{u \in X: u(k) \geq 0, \Delta u(k) \text{ is a decreasing function for } k \in \mathbb{Z}_{0, T+1}\}$. Obviously, $K, K' \subset X$ are two cones with $K' \subset K$. For $u \in K$, we define

$$(Au)(k) = \sum_{j=1}^T G(k, j)f(j, u(j)), \quad k \in \mathbb{Z}_{0, T+1}, \quad (2.6)$$

$$(\Phi u)(k) = [(Au)(k)]^+, \quad k \in \mathbb{Z}_{0, T+1}, \quad (2.7)$$

$$\varphi(u) = \min_{k \in \{h, \dots, T+1-h\}} u(k), \quad (2.8)$$

where $(B)^+ = \max\{B, 0\}$. For $u \in K'$, define

$$(\Phi' u)(k) = \sum_{j=1}^T G(k, j) f^+(j, u(j)), \quad k \in \mathbb{Z}_{0, T+1}, \quad (2.9)$$

where $f^+(j, u) = \max\{f(j, u), 0\}$.

By the above definition, it is easy to see that $A: K \rightarrow X$ and $\Phi: K \rightarrow K$. Moreover, we have the following lemma.

Lemma 2.3. $\Phi': K' \rightarrow K'$.

Proof. Let $u \in K'$. By (2.9), it is obvious that $(\Phi' u)(k) \geq 0$. From (2.3) and (2.9), we have

$$\begin{aligned} (\Phi' u)(k) &= - \sum_{j=1}^{k-1} (k-j) f^+(j, u(j)) + \sum_{j=1}^T \frac{(k+\beta)(T+1-j) f^+(j, u(j))}{(T+1-\alpha l) + \beta(1-\alpha)} \\ &\quad - \sum_{j=1}^{l-1} \frac{\alpha(k+\beta)(l-j) f^+(j, u(j))}{(T+1-\alpha l) + \beta(1-\alpha)}, \end{aligned}$$

and so for all $k \in \mathbb{Z}_{0, T+1}$,

$$\begin{aligned} \Delta(\Phi' u)(k) &= - \sum_{j=1}^k f^+(j, u(j)) + \sum_{j=1}^T \frac{(T+1-j) f^+(j, u(j))}{(T+1-\alpha l) + \beta(1-\alpha)} \\ &\quad - \sum_{j=1}^{l-1} \frac{\alpha(l-j) f^+(j, u(j))}{(T+1-\alpha l) + \beta(1-\alpha)}. \end{aligned}$$

Then $\nabla \Delta(\Phi' u)(k) = -f^+(k, u(k)) < 0$, so $\Delta(\Phi' u)(k)$ is decreasing. \square

Theorem 2.1. Suppose $f(k, 0) \geq 0$ and $f(k, 0) \neq 0$ for $k \in \mathbb{Z}_{0, T+1}$, and there exist nonnegative numbers a, b and d such that

$$0 < \left(1 + \frac{T+1}{\beta}\right) \frac{T+1-\alpha l}{\alpha(T+1-l)} d < a < \frac{h}{T+1} b < b.$$

If f satisfies the following conditions:

- (H1) $f(k, u) \geq 0$ for $(k, u) \in \mathbb{Z}_{0, T+1} \times [d, b]$,
- (H2) $f(k, u) < \frac{a}{M}$ for $(k, u) \in \mathbb{Z}_{0, T+1} \times [0, a]$,
- (H3) $f(k, u) > \frac{b}{m}$ for $(k, u) \in \mathbb{Z}_{0, T+1} \times [\frac{h}{T+1} b, b]$,

then BVP $\{(1.1), (1.2)\}$ has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a \leq \|u_2\|, \quad \varphi(u_2) < \frac{h}{T+1} b.$$

Proof. Let $X = C(\mathbb{Z}_{0, T+1})$, and $K, K', \Phi, \Phi', \varphi$ be defined as above. From the continuity of f and Lemma 2.3, it is clear that $A: K \rightarrow X$, $\Phi: K \rightarrow K$ and $\Phi': K' \rightarrow K'$ are completely continuous. Moreover, for $u \in K'$, we have $\min_{k \in \{h, \dots, T+1-h\}} u(k) \geq \frac{h}{T+1} \|u\|$ because $\Delta u(k)$ is

a decreasing function. Therefore, $\varphi(u) \leq \|u\| \leq \frac{T+1}{h}\varphi(u)$. We will verify (C1)–(C3) of Theorem A.

For any $u \in \partial K_a$, from (H2) we obtain

$$\begin{aligned}\|\Phi u\| &= \max_{k \in \mathbb{Z}_{0,T+1}} \left[\sum_{j=1}^T G(k, j) f(j, u(j)) \right]^+ \\ &= \max_{k \in \mathbb{Z}_{0,T+1}} \max \left\{ \sum_{j=1}^T G(k, j) f(j, u(j)), 0 \right\} \\ &< \max_{k \in \mathbb{Z}_{0,T+1}} \frac{a}{M} \sum_{j=1}^T G(k, j) = a.\end{aligned}$$

Hence (C1) holds.

Next we show that (C2) of Theorem A is satisfied. For $u \in \partial K'_a$, i.e., $\|u\| = a$, we have from (H2)

$$\|\Phi' u\| = \max_{k \in \mathbb{Z}_{0,T+1}} \sum_{j=1}^T G(k, j) f^+(j, u(j)) < \max_{k \in \mathbb{Z}_{0,T+1}} \frac{a}{M} \sum_{j=1}^T G(k, j) = a.$$

Next, let $u \in \partial K'(\frac{h}{T+1}b)$, i.e., $\varphi(u) = \frac{h}{T+1}b$. For $k \in \{h, h+1, \dots, T+1-h\}$, we have $\frac{h}{T+1}b \leq u(k) \leq b$. With condition (H3), we get

$$\begin{aligned}\varphi(\Phi' u) &= \min_{k \in \{h, h+1, \dots, T+1-h\}} \sum_{j=1}^T G(k, j) f^+(j, u(j)) \\ &\geq \min_{k \in \{h, h+1, \dots, T+1-h\}} \sum_{j=h}^{T+1-h} G(k, j) f^+(j, u(j)) \\ &> \min_{k \in \{h, h+1, \dots, T+1-h\}} \frac{b}{m} \sum_{j=h}^{T+1-h} G(k, j) \\ &= b \geq \frac{h}{T+1}b.\end{aligned}$$

Finally, we show that (C3) of Theorem A is also satisfied. For $u \in K'_a(\frac{h}{T+1}b) \cap \{u: \Phi' u = u\}$, we have $\|u\| > a > (1 + \frac{T+1}{\beta}) \frac{T+1-\alpha l}{\alpha(T+1-l)}d$. First we claim that

$$u(0) \geq \frac{T+1-\alpha l}{\alpha(T+1-l)}d.$$

If not, then $u(0) < \frac{T+1-\alpha l}{\alpha(T+1-l)}d$. Suppose $u(k_1) = \|u\|$, then

$$u(k_1) - u(0) > \frac{T+1}{\beta} \frac{T+1-\alpha l}{\alpha(T+1-l)}d,$$

i.e.,

$$\Delta u(0) + \dots + \Delta u(k_1 - 1) > \frac{T+1}{\beta} \frac{T+1-\alpha l}{\alpha(T+1-l)}d.$$

Then there exists $k_0 \in \{0, \dots, k_1\}$ such that

$$\Delta u(k_0) > \frac{T+1}{\beta k_1} \frac{T+1-\alpha l}{\alpha(T+1-l)} d.$$

Again by (1.2),

$$u(0) = \beta \Delta u(0) \geq \beta \Delta u(k_0) > \frac{T+1-\alpha l}{\alpha(T+1-l)} d,$$

a contradiction. Next, observe that we must have $u(T+1) \geq d$, for if it is not true, by the assumption that $\Delta u(k)$ is decreasing, we have $(T+1)u(l) - lu(T+1) \geq (T+1-l)u(0)$. From $u(T+1) = \alpha u(l)$, we have

$$u(0) \leq \frac{T+1-\alpha l}{\alpha(T+1-l)} u(T+1) < \frac{T+1-\alpha l}{\alpha(T+1-l)} d.$$

This contradicts with $u(0) \geq \frac{T+1-\alpha l}{\alpha(T+1-l)} d$. Therefore, $d \leq u(k) \leq b$ for $k \in \mathbb{Z}_{0,T+1}$. Condition (H1) now ensures that $f^+(k, u(k)) = f(k, u(k))$, hence $\Phi u = \Phi' u$ for $u \in K'_a(\frac{h}{T+1}b) \cap \{u: \Phi' u = u\}$. An application of Theorem A now asserts that Φ has in K at least two fixed points u_1 and u_2 such that

$$0 \leq \|u_1\| < a < \|u_2\|, \quad \varphi(u_2) < \frac{h}{T+1} b.$$

Now we will show that these two fixed points u_i , $i = 1, 2$, of Φ , are also fixed points of A . Suppose u_i is not a fixed point of A , then there exists $k_0 \in \mathbb{Z}_{0,T+1}$ such that $u_i(k_0) \neq (Au_i)(k_0)$. So $(Au_i)(k_0) < 0 < u_i(k_0)$. Let \mathbb{Z}_{k_1,k_2} be the largest set of consecutive integers such that $k_0 \in \mathbb{Z}_{k_1,k_2}$ and $(Au_i)(k) < 0$, $k \in \mathbb{Z}_{k_1,k_2}$. From the assumption $f(k, 0) \geq 0$ ($\neq 0$) for $k \in \mathbb{Z}_{0,T+1}$, we have $\mathbb{Z}_{k_1,k_2} \neq \mathbb{Z}_{0,T+1}$. If $k_1 > 0$, then $\Delta Au_i(k_1 - 1) = Au_i(k_1) - Au_i(k_1 - 1) < 0$ by $Au_i(k_1) < 0$ and $Au_i(k_1 - 1) > 0$. From $\Delta Au_i(k_1) - \Delta Au_i(k_1 - 1) = \nabla \Delta Au_i(k_1) = -f(k_1, 0) \leq 0$, we have $\Delta Au_i(k_1) \leq \Delta Au_i(k_1 - 1) < 0$. In the same way, we can get $\Delta Au_i(k) < 0$, $k \in \mathbb{Z}_{k_1,k_2}$. Thus $0 > Au_i(k_1) > Au_i(k_1 + 1) > \dots > Au_i(k_2)$ and $k_2 = T + 1$, i.e., $Au_i(T + 1) < Au_i(l)$. On the other hand, $Au_i(T + 1) = \alpha Au_i(l)$ and $0 < \alpha < 1$ mean that $Au_i(T + 1) > Au_i(l)$, this contradicts with $Au_i(T + 1) < Au_i(l)$. If $k_1 = 0$, from $(Au_i)(0) - \beta \Delta(Au_i)(0) = 0$, we have $\Delta Au_i(0) < 0$. In a similar way, we can get $k_2 = T + 1$, and $Au_i(T + 1) < Au_i(l)$, a contradiction. Hence u_i must be a fixed point of A , $i = 1, 2$. Obviously, u_i is a solution of BVP $\{(1.1), (1.2)\}$ if and only if u_i is a fixed point of A . Therefore, BVP $\{(1.1), (1.2)\}$ has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a \leq \|u_2\|, \quad \varphi(u_2) < \frac{h}{T+1} b. \quad \square$$

Similarly, applying Theorem A repeatedly, we easily get the following results.

Theorem 2.2. Suppose $f(k, 0) \geq 0$ and $f(k, 0) \neq 0$ for $k \in \mathbb{Z}_{0,T+1}$, and there exist nonnegative numbers $a_i, b_i, d_i > 0$, $i = 1, 2, \dots, n$, satisfying

$$a_1 < b_1 < a_2 < \dots < a_n < b_n, \\ \left(1 + \frac{T+1}{\beta}\right) \frac{T+1-\alpha l}{\alpha(T+1-l)} d_i < a_i < \frac{h_i}{T+1} b_i < b_i, \quad d_i \leq d_{i+1}.$$

If f satisfies the following conditions:

- (H4) $f(k, u) \geq 0$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [d_i, b_i]$,
 (H5) $f(k, u) < \frac{a_i}{M}$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [0, a_i]$,
 (H6) $f(k, u) > \frac{b_i}{m}$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [\frac{h_i}{T+1}b_i, b_i]$, $h_i \in \{2, \dots, [T/2]\}$ for each i ,

then BVP $\{(1.1)–(1.2)\}$ has at least $n + 1$ positive solutions u_1, u_2, \dots, u_{n+1} satisfying

$$\begin{aligned} 0 \leq \|u_1\| < a_1 < \|u_2\|, \quad \varphi(u_2) < \frac{h_1}{T+1}b_1, \\ a_2 < \|u_3\|, \quad \varphi(u_3) < \frac{h_2}{T+1}b_2, \\ \vdots \\ a_n < \|u_{n+1}\|, \quad \varphi(u_{n+1}) < \frac{h_n}{T+1}b_n. \end{aligned}$$

Theorem 2.3. Suppose $f(k, 0) \geq 0$ and $f(k, 0) \not\equiv 0$ for $k \in \mathbb{Z}_{0,T+1}$, and there exist nonnegative numbers $a_i, b_i > 0$, $i = 1, 2, \dots, n$, and $d > 0$ satisfying

$$\left(1 + \frac{T+1}{\beta}\right) \frac{T+1-\alpha l}{\alpha(T+1-l)} d \leq a_1, \quad b_{i-1} \leq a_i < \frac{h_i}{T+1}b_i < b_i.$$

If f satisfies the following conditions:

- (H7) $f(k, u) \geq 0$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [d, b_n]$,
 (H8) $f(k, u) < \frac{a_i}{M}$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [0, a_i]$,
 (H9) $f(k, u) > \frac{b_i}{m}$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [\frac{h_i}{T+1}b_i, b_i]$,

then BVP $\{(1.1), (1.2)\}$ has at least $2n$ positive solutions u_1, u_2, \dots, u_{2n} satisfying

$$\begin{aligned} 0 \leq \|u_1\| < a_1 < \|u_2\|, \quad \varphi(u_2) < \frac{h_1}{T+1}b_1 < \varphi(u_3), \\ \|u_3\| < a_2 < \|u_4\|, \quad \varphi(u_4) < \frac{h_2}{T+1}b_2 < \varphi(u_5), \\ \vdots \\ \|u_{2n-1}\| < a_n < \|u_{2n}\|, \quad \varphi(u_{2n}) < \frac{h_n}{T+1}b_n. \end{aligned}$$

Remark 2.1. In a similar way, we can get the corresponding results for the difference equation (1.1) subject to the boundary conditions

$$u(0) = \alpha u(l), \quad u(T+1) + \beta \nabla u(T+1) = 0, \quad (2.10)$$

which are symmetric to the boundary conditions (1.2).

Remark 2.2. The results can also be generalized to a discrete m -point boundary value problem, that is, BVPs of the form

$$\begin{cases} \nabla \Delta u(k) + f(k, u(k)) = 0, & k \in N, \\ u(0) - \beta \Delta u(0) = 0, & u(T+1) = \sum_{i=1}^{m-2} \alpha_i u(l), \end{cases} \quad (2.11)$$

$$(2.12)$$

and

$$\begin{cases} \nabla \Delta u(k) + f(k, u(k)) = 0, & k \in N, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(l), & u(T+1) + \beta \nabla u(T+1) = 0. \end{cases} \quad (2.13)$$

$$(2.14)$$

3. Positive solutions to BVP {(1.1), (1.3)}

Lemma 3.1. Suppose that $1 - \alpha \neq 0$, $y \in C(\mathbb{Z}_{1,T})$, then the BVP

$$\begin{cases} \nabla \Delta u(k) + y(k) = 0, & k \in \mathbb{Z}_{1,T}, \\ \Delta u(0) = 0, & u(T+1) = \alpha u(l) \end{cases} \quad (3.1)$$

$$(3.2)$$

has a unique solution

$$u(k) = -\sum_{j=1}^{k-1} (k-j)y(j) + \sum_{j=1}^T \frac{T+1-j}{1-\alpha} y(j) - \sum_{j=1}^{l-1} \frac{\alpha(l-j)}{1-\alpha} y(j), \quad k \in \mathbb{Z}_{0,T+1}. \quad (3.3)$$

Proof. From (3.1) we have, for any $i \in \mathbb{Z}_{1,T}$,

$$\Delta u(i) - \Delta u(i-1) = -y(i).$$

Then for any $j \in \mathbb{Z}_{1,T}$,

$$\Delta u(j) - \Delta u(0) = \sum_{i=1}^j [\Delta u(i) - \Delta u(i-1)] = -\sum_{i=1}^j y(i).$$

Since $\Delta u(0) = 0$, we have

$$\Delta u(j) = -\sum_{i=1}^j y(i).$$

So

$$u(k) = \sum_{j=1}^{k-1} \Delta u(j) + u(1) = -\sum_{j=1}^{k-1} \sum_{i=1}^j y(i) + u(1) = -\sum_{j=1}^{k-1} (k-j)y(j) + u(1).$$

Together with the boundary condition $u(T+1) = \alpha u(l)$, we get

$$u(1) = \sum_{j=1}^T \frac{T+1-j}{1-\alpha} y(j) - \sum_{j=1}^{l-1} \frac{\alpha(l-j)}{1-\alpha} y(j).$$

So

$$u(k) = -\sum_{j=1}^{k-1} (k-j)y(j) + \sum_{j=1}^T \frac{T+1-j}{1-\alpha} y(j) - \sum_{j=1}^{l-1} \frac{\alpha(l-j)}{1-\alpha} y(j), \quad k \in \mathbb{Z}_{0,T+1}. \quad \square$$

Lemma 3.2. Suppose $1 - \alpha \neq 0$, then the Green's function for the BVP $\{(3.1), (3.2)\}$ is given by

$$\widehat{G}(k, j) = \begin{cases} \frac{(T+1-k)+\alpha(k-l)}{1-\alpha}, & j < k, \quad j < l, \\ \frac{(T+1-k)+\alpha(k-j)}{1-\alpha}, & l \leq j < k, \\ \frac{(T+1-j)+\alpha(j-l)}{1-\alpha}, & k \leq j < l, \\ \frac{T+1-j}{1-\alpha}, & j \geq k, \quad j \geq l. \end{cases} \quad (3.4)$$

Proof. If $k \geq l$, the unique solution (3.3) can be written as

$$\begin{aligned} u(k) &= -\sum_{j=1}^{l-1} (k-j)y(j) - \sum_{j=l}^{k-1} (k-j)y(j) + \sum_{j=1}^{l-1} \frac{T+1-j}{1-\alpha} y(j) + \sum_{j=l}^{k-1} \frac{T+1-j}{1-\alpha} y(j) \\ &\quad + \sum_{j=k}^T \frac{T+1-j}{1-\alpha} y(j) - \sum_{j=1}^{l-1} \frac{\alpha(l-j)}{1-\alpha} y(j) \\ &= \sum_{j=1}^{l-1} \frac{(T+1-k)+\alpha(k-l)}{1-\alpha} y(j) + \sum_{j=l}^{k-1} \frac{(T+1-k)+\alpha(k-j)}{1-\alpha} y(j) \\ &\quad + \sum_{j=k}^T \frac{T+1-j}{1-\alpha} y(j). \end{aligned}$$

Similarly, if $k < l$, the unique solution (3.3) can be written as

$$\begin{aligned} u(k) &= \sum_{j=1}^{k-1} \frac{(T+1-k)+\alpha(k-l)}{1-\alpha} y(j) + \sum_{j=k}^{l-1} \frac{(T+1-j)+\alpha(j-l)}{1-\alpha} y(j) \\ &\quad + \sum_{j=l}^T \frac{T+1-j}{1-\alpha} y(j). \end{aligned}$$

Together with the fact that the unique solution of $\{(3.1), (3.2)\}$ can be written as $u(k) = \sum_{j=1}^T \widehat{G}(k, j)y(j)$, we obtain (3.4). \square

By our basic assumption, $l \in \mathbb{Z}_{2, T-1}$ and $0 < \alpha < 1$, it is clear that the Green's function $\widehat{G}(k, j)$ for the BVP $\{(3.1), (3.2)\}$ as given in (3.4) satisfies $\widehat{G}(k, j) > 0$ for $k \in \mathbb{Z}_{0, T+1}$, $j \in \mathbb{Z}_{1, T}$. Letting $y(k) \equiv 1$ in (3.3), we get

$$\sum_{j=1}^T \widehat{G}(k, j) = \begin{cases} -\frac{k(k-1)}{2} + \frac{T(T+1)-\alpha l(l-1)}{2(1-\alpha)}, & k \in \mathbb{Z}_{1, T+1}, \\ \frac{T(T+1)-\alpha l(l-1)}{2(1-\alpha)}, & k = 0. \end{cases} \quad (3.5)$$

Let

$$\widehat{M} = \max_{k \in \mathbb{Z}_{0, T+1}} \sum_{j=1}^T \widehat{G}(k, j) \quad \text{and} \quad \widehat{m} = \min_{k \in \{h, \dots, T+1-h\}} \sum_{j=h}^{T+1-h} \widehat{G}(k, j),$$

here h is a fixed integer in $\{2, \dots, [\frac{T}{2}]\}$. It is clear that $0 < \widehat{m} < \widehat{M}$.

Let $X = C(\mathbb{Z}_{0,T+1})$, $\widehat{K} = \{u \in X : u(k) \geq 0, k \in \mathbb{Z}_{0,T+1}\}$, $\widehat{K}' = \{u \in X : u(k) \geq 0 \text{ is a decreasing concave function for } k \in \mathbb{Z}_{0,T+1}\}$. Obviously, $\widehat{K}, \widehat{K}' \subset X$ are two cones with $\widehat{K}' \subset \widehat{K}$. For $u \in \widehat{K}$, we define

$$(\widehat{A}u)(k) = \sum_{j=1}^T \widehat{G}(k, j) f(j, u(j)), \quad k \in \mathbb{Z}_{0,T+1}, \quad (3.6)$$

$$(\widehat{\Phi}u)(k) = [(\widehat{A}u)(k)]^+, \quad k \in \mathbb{Z}_{0,T+1}, \quad (3.7)$$

$$\varphi(u) = \min_{k \in \{h, \dots, T+1-h\}} u(k), \quad (3.8)$$

where $(B)^+ = \max\{B, 0\}$. For $u \in \widehat{K}'$, define

$$(\widehat{\Phi}'u)(k) = \sum_{j=1}^T \widehat{G}(k, j) f^+(j, u(j)), \quad k \in \mathbb{Z}_{0,T+1}, \quad (3.9)$$

where $f^+(j, u) = \max\{f(j, u), 0\}$.

Remark 3.1. By the above definition, it is easy to see that $\widehat{A} : \widehat{K} \rightarrow X$ and $\widehat{\Phi} : \widehat{K} \rightarrow \widehat{K}$. Moreover, we have $\widehat{\Phi}' : \widehat{K}' \rightarrow \widehat{K}'$. In fact, By (3.9), we easily see that $\widehat{\Phi}'u(k) \geq 0$. From the definition of $\widehat{\Phi}'u$ and the fact that $\widehat{G}(k, j)$ is decreasing with respect to k , we see that $\widehat{\Phi}'u$ is decreasing. By (3.3) and (3.9), we derive that $\Delta(\widehat{\Phi}'u)(k) = -\sum_{j=1}^k f^+(j, u(j))$ is decreasing, hence $\widehat{\Phi}'u$ is concave. Therefore, $\widehat{\Phi}'u$ is a nonnegative decreasing concave function for $k \in \mathbb{Z}_{0,T+1}$.

Theorem 3.1. Suppose $f(k, 0) \geq 0$ and $f(k, 0) \neq 0$ for $k \in \mathbb{Z}_{0,T+1}$, and there exist nonnegative numbers a, b and d such that

$$0 < d < \frac{T+1-\alpha l}{\alpha(T+1-l)} d < a < \frac{T+1-h}{T+1} b < b.$$

If f satisfies the following conditions:

- (H1) $^\wedge$ $f(k, u) \geq 0$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [d, b]$,
- (H2) $^\wedge$ $f(k, u) < \frac{a}{M}$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [0, a]$,
- (H3) $^\wedge$ $f(k, u) > \frac{b}{m}$ for $(k, u) \in \mathbb{Z}_{0,T+1} \times [\frac{T+1-h}{T+1}b, b]$,

then BVP $\{(1.1), (1.3)\}$ has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a \leq \|u_2\|, \quad \varphi(u_2) < \frac{T+1-h}{T+1} b.$$

Proof. Let $X = C(\mathbb{Z}_{0,T+1})$, and $\widehat{K}, \widehat{K}', \widehat{\Phi}, \widehat{\Phi}', \varphi$ be defined as above. From the continuity of f and Remark 3.1, it is clear that $\widehat{A} : \widehat{K} \rightarrow X$, $\widehat{\Phi} : \widehat{K} \rightarrow \widehat{K}$ and $\widehat{\Phi}' : \widehat{K}' \rightarrow \widehat{K}'$ are completely continuous. Moreover, for $u \in \widehat{K}'$, we claim that $\varphi(u) \leq u(0) = \|u\| \leq \frac{T+1}{T+1-h} \varphi(u)$.

In fact, by assumption, $u(k)$ is decreasing and concave, so we have

$$\frac{u(0) - u(T+1)}{T+1} \leq \frac{u(k) - u(T+1)}{T+1-k}.$$

Then

$$\frac{u(0)}{T+1} \leq \frac{u(k)}{T+1-k}.$$

Hence

$$\|u\| = u(0) \leq \frac{T+1}{T+1-h} u(k).$$

Take the minimum for $k \in \{h, h+1, \dots, T+1-h\}$ on both sides, we have

$$\|u\| = u(0) \leq \frac{T+1}{T+1-h} \varphi(u).$$

Since $u(k)$ is a decreasing function, we get $\varphi(u) \leq u(0) = \|u\| \leq \frac{T+1}{T+1-h} \varphi(u)$.

Similar to the proof of Theorem 2.1, we will verify (C1)–(C3) of Theorem A. Now as (C1) and (C2) can be established by exactly the same arguments as those in the proof of Theorem 2.1, we will skip these here. For (C3), for any $u \in \widehat{K}'_a(\frac{T+1-h}{T+1}b) \cap \{u: \widehat{\Phi}'u = u\}$, we have $u(0) = \|u\| > a > \frac{T+1-\alpha l}{\alpha(T+1-l)}d$. Observe that $u(T+1) \geq d$. For if not, since $u(k)$ is decreasing and concave, we have

$$\frac{u(l) - u(T+1)}{T+1-l} \geq \frac{u(0) - u(T+1)}{T+1},$$

i.e., $(T+1)u(l) - lu(T+1) \geq (T+1-l)u(0)$. From $u(T+1) = \alpha u(l)$, we have

$$u(T+1) \geq \frac{\alpha(T+1-l)}{T+1-\alpha l} u(0) > d.$$

Therefore, by the decreasing property of $u(k)$, we get $d \leq u(k) \leq b$ for $k \in \mathbb{Z}_{0,T+1}$. Condition (H1)[^] now ensures that $f^+(k, u(k)) = f(k, u(k))$, hence $\widehat{\Phi}u = \widehat{\Phi}'u$ for $u \in \widehat{K}'_a(\frac{T+1-h}{T+1}b) \cap \{u: \widehat{\Phi}'u = u\}$. An application of Theorem A now asserts that $\widehat{\Phi}$ has in \widehat{K} at least two fixed points u_1 and u_2 such that

$$0 \leq \|u_1\| < a < \|u_2\|, \quad \varphi(u_2) < \frac{T+1-h}{T+1}b.$$

We will show that these two fixed point u_i , $i = 1, 2$, of $\widehat{\Phi}$ are also fixed points of \widehat{A} . Suppose u_i is not a fixed point of \widehat{A} , then there exists $k_0 \in \mathbb{Z}_{0,T+1}$ such that $u_i(k_0) \neq (\widehat{A}u_i)(k_0)$. So $(\widehat{A}u_i)(k_0) < 0 < u_i(k_0)$. Let \mathbb{Z}_{k_1,k_2} be the largest set of consecutive integers such that $k_0 \in \mathbb{Z}_{k_1,k_2}$ and $(\widehat{A}u_i)(k) < 0$, $k \in \mathbb{Z}_{k_1,k_2}$. By the assumption $f(k, 0) \geq 0 (\neq 0)$ for $k \in \mathbb{Z}_{0,T+1}$, we have $\mathbb{Z}_{k_1,k_2} \not\subseteq \mathbb{Z}_{0,T+1}$. From the boundary condition $\Delta u(0) = 0$, we know that $k_1 > 0$, then $\Delta \widehat{A}u_i(k_1 - 1) = \widehat{A}u_i(k_1) - \widehat{A}u_i(k_1 - 1) < 0$ since $\widehat{A}u_i(k_1) < 0$ and $\widehat{A}u_i(k_1 - 1) > 0$. From $\Delta \widehat{A}u_i(k_1) - \Delta \widehat{A}u_i(k_1 - 1) = \nabla \Delta \widehat{A}u_i(k_1) = -f(k_1, 0) \leq 0$, we have $\Delta \widehat{A}u_i(k_1) \leq \Delta \widehat{A}u_i(k_1 - 1) < 0$. In the same way, we can get $\Delta \widehat{A}u_i(k) < 0$, $k \in \mathbb{Z}_{k_1,k_2}$. Thus $0 > \widehat{A}u_i(k_1) > \widehat{A}u_i(k_1 + 1) > \dots > \widehat{A}u_i(k_2)$ and $k_2 = T+1$, i.e., $\widehat{A}u_i(T+1) < \widehat{A}u_i(l)$. On the other hand, $\widehat{A}u_i(T+1) = \alpha \widehat{A}u_i(l)$ and $0 < \alpha < 1$ mean that $\widehat{A}u_i(T+1) > \widehat{A}u_i(l)$, this contradicts with $\widehat{A}u_i(T+1) < \widehat{A}u_i(l)$. Hence u_i must be a fixed point of \widehat{A} , $i = 1, 2$. Obviously, u_i is a solution of BVP $\{(1.1), (1.3)\}$ if and only if u_i is a fixed point of \widehat{A} . Therefore, BVP $\{(1.1), (1.3)\}$ has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a \leq \|u_2\|, \quad \varphi(u_2) < \frac{T+1-h}{T+1}b. \quad \square$$

Theorem 3.2. Suppose $f(k, 0) \geq 0$ and $f(k, 0) \neq 0$ for $k \in \mathbb{Z}_{0,T+1}$, and there exist nonnegative numbers $a_i, b_i, d_i > 0$, $i = 1, 2, \dots, n$, satisfying

$$a_1 < b_1 < a_2 < \dots < a_n < b_n, \\ \frac{T+1-\alpha l}{\alpha(T+1-l)}d_i < a_i < \frac{T+1-h_i}{T+1}b_i < b_i, \quad d_i \leq d_{i+1}.$$

If f satisfies the following conditions:

$$(H4)^\wedge \quad f(k, u) \geq 0 \text{ for } (k, u) \in \mathbb{Z}_{0,T+1} \times [d_i, b_i],$$

$$(H5)^\wedge \quad f(k, u) < \frac{a_i}{M} \text{ for } (k, u) \in \mathbb{Z}_{0,T+1} \times [0, a_i],$$

$$(H6)^\wedge \quad f(k, u) > \frac{b_i}{m} \text{ for } (k, u) \in \mathbb{Z}_{0,T+1} \times [\frac{T+1-h_i}{T+1}b_i, b_i], \quad h_i \in \{2, \dots, [T/2]\} \text{ for each } i,$$

then BVP $\{(1.1), (1.3)\}$ has at least $n + 1$ positive solutions u_1, u_2, \dots, u_{n+1} satisfying

$$0 \leq \|u_1\| < a_1 < \|u_2\|, \quad \varphi(u_2) < \frac{T+1-h_1}{T+1}b_1,$$

$$a_2 < \|u_3\|, \quad \varphi(u_3) < \frac{T+1-h_2}{T+1}b_2,$$

\vdots

$$a_n < \|u_{n+1}\|, \quad \varphi(u_{n+1}) < \frac{T+1-h_n}{T+1}b_n.$$

Theorem 3.3. Suppose $f(k, 0) \geq 0$ and $f(k, 0) \neq 0$ for $k \in \mathbb{Z}_{0,T+1}$, and there exist nonnegative numbers $a_i, b_i > 0$, $i = 1, 2, \dots, n$, and $d > 0$ satisfying

$$\frac{T+1-\alpha l}{\alpha(T+1-l)}d \leq a_1, \quad b_{i-1} \leq a_i < \frac{T+1-h_i}{T+1}b_i < b_i.$$

If f satisfies the following conditions:

$$(H7)^\wedge \quad f(k, u) \geq 0 \text{ for } (k, u) \in \mathbb{Z}_{0,T+1} \times [d, b_n],$$

$$(H8)^\wedge \quad f(k, u) < \frac{a_i}{M} \text{ for } (k, u) \in \mathbb{Z}_{0,T+1} \times [0, a_i],$$

$$(H9)^\wedge \quad f(k, u) > \frac{b_i}{m} \text{ for } (k, u) \in \mathbb{Z}_{0,T+1} \times [\frac{T+1-h_i}{T+1}b_i, b_i], \quad h_i \in \{2, \dots, [T/2]\} \text{ for each } i,$$

then BVP $\{(1.1), (1.3)\}$ has at least $2n$ positive solutions u_1, u_2, \dots, u_{2n} satisfying

$$0 \leq \|u_1\| < a_1 < \|u_2\|, \quad \varphi(u_2) < \frac{T+1-h_1}{T+1}b_1 < \varphi(u_3),$$

$$\|u_3\| < a_2 < \|u_4\|, \quad \varphi(u_4) < \frac{T+1-h_2}{T+1}b_2 < \varphi(u_5),$$

\vdots

$$\|u_{2n-1}\| < a_n < \|u_{2n}\|, \quad \varphi(u_{2n}) < \frac{T+1-h_n}{T+1}b_n.$$

Remark 3.2. In a similar way, we can get the corresponding results for the difference equation (1.1) subject to the boundary conditions

$$u(0) = \alpha u(l), \quad \Delta u(T+1) = 0, \tag{3.10}$$

which are symmetric to the boundary conditions (1.3).

Remark 3.3. The results can also be generalized to a discrete m -point boundary value problem, that is, BVPs of the form

$$\begin{cases} \nabla \Delta u(k) + f(k, u(k)) = 0, & k \in N, \end{cases} \quad (3.11)$$

$$\begin{cases} \Delta u(0) = 0, & u(T+1) = \sum_{i=1}^{m-2} \alpha_i u(i), \end{cases} \quad (3.12)$$

and

$$\begin{cases} \nabla \Delta u(k) + f(k, u(k)) = 0, & k \in N, \end{cases} \quad (3.13)$$

$$\begin{cases} u(0) = \sum_{i=1}^{m-2} \alpha_i u(i), & \Delta u(T+1) = 0. \end{cases} \quad (3.14)$$

Remark 3.4. The assumptions that “ $\alpha \neq 0$ ” and “ $\beta \neq 0$ ” are essential in the proofs above and cannot be removed. For “ $\alpha = 0$ ” or “ $\beta = 0$,” the method in this paper is not valid. We will discuss these cases in a subsequent paper.

4. Application

We consider the following BVP:

$$\nabla \Delta u(k) + f(k, u(k)) = 0, \quad k \in \mathbb{Z}_{1,4}, \quad (4.1)$$

$$\Delta u(0) = 0, \quad u(5) = \frac{3}{5}u(3), \quad (4.2)$$

where

$$f(k, u) = \begin{cases} \frac{3}{2}u - \frac{445}{14}, & 22 \leq u \leq 40, \\ \frac{3}{14}u - \frac{7}{2}, & 21 \leq u \leq 22, \\ 1, & 13 \leq u \leq 21, \\ u - 12, & 12 \leq u \leq 13, \\ -\frac{1}{5}u + \frac{12}{5}, & 7 \leq u \leq 12, \\ \frac{1}{8}(u-3)^2 - 1, & 0 \leq u \leq 7. \end{cases} \quad (4.3)$$

From (3.4) and (3.5), we have

$$\widehat{M} = \max_{k \in \mathbb{Z}_{0,T+1}} \sum_{j=1}^T \widehat{G}(k, j) = 20.5, \quad \widehat{m} = \min_{k \in \{h, \dots, T+1-h\}} \sum_{j=h}^{T+1-h} \widehat{G}(k, j) = 10,$$

here $h = 2$.

Let $a = 21$, $b = 40$, $d = 7$, we easily see that all the assumptions of Theorem 3.1 are satisfied. So BVP $\{(4.1), (4.2)\}$ has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < a \leq \|u_2\|, \quad \varphi(u_2) < \frac{T+1-h}{T+1}b.$$

In fact, it is easy to verify that

$$u_1(0) = 18, \quad u_1(1) = 18, \quad u_1(2) = 17, \quad u_1(3) = 15, \quad u_1(4) = 12, \quad u_1(5) = 9$$

and

$$u_2(0) = 22, \quad u_2(1) = 22, \quad u_2(2) = \frac{291}{14}, \quad u_2(3) = \frac{130}{7}, \quad u_2(4) = \frac{215}{14}, \\ u_2(5) = \frac{78}{7}$$

are two positive solutions, satisfying

$$0 < \|u_1\| < 21 \leq \|u_2\|, \quad \varphi(u_2) < \frac{3}{5}b = 24.$$

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